

The matching relaxation for a class of generalized set partitioning problems

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Abstract

This paper introduces a discrete relaxation for the class of combinatorial optimization problems which can be described by a set partitioning formulation under packing constraints. We present two combinatorial relaxations based on computing maximum weighted matchings in suitable graphs. Besides providing dual bounds, the relaxations are also used on a variable reduction technique and a matheuristic. We show how that general method can be tailored to sample applications and, in particular, perform a successful computational evaluation with benchmark instances of a problem in maritime logistics.

Keywords: dual bounds, matchings, set partitioning, integer programming.

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1 Introduction

Consider the following problem, which the reader might recognize from a range of application domains. A set of n tasks is to be performed, and the problem solver has to decide for an execution mode to them. To each individual task corresponds a particular set of acceptable execution modes to perform it. Moreover, each execution mode comprises a well-defined cost and resource usage (possibly many, *e.g.* time, space, tools, human workers). Since the finite resources are to be shared among tasks, the problem solver also counts on an oracle capable of determining in constant time whether a selection of execution modes to different tasks is *compatible*. Any solution prescribing the assignment of incompatible execution modes is thus rendered infeasible.

We concern the class of problems which, at its core, can be cast as follows. Given sets of acceptable assignments to each individual task, and compatibility information among any selection of assignments, find a compatible setting for the n tasks of minimum total cost. To clearly outline our contributions and the scope of our investigation, we first give a precise formulation of the class of problems we concern, which we refer to as the generalized set partitioning problem (GSPP).

The base GSPP Let T be the set of tasks, and $R = R_1 \times R_2 \times \dots \times R_k$ be the set of tuples identifying resource usage, *i.e.* combinations of an option for resource R_1 , an option for resource R_2 , and so forth. Also define Ω_i as the set of feasible assignments to task i : each element in Ω_i assigns a subset of resources in R (characterizing an execution mode) to complete an individual task. We denote $\Omega = \{x : x \in \Omega_i, i = 1, \dots, n\}$. Binary decision variables $\mathbf{y} \in \mathbb{B}^{|\Omega|}$ thus indicate which individual assignments are used in the solution. Let $c_j \in \mathbb{Q}$ denote the cost of an assignment y_j to an individual task, consisting of a computable function of the total resource usage on that individual assignment alone. Finally, the coefficient matrices are as follows. $A \in \mathbb{B}^{|T| \times |\Omega|}$ associates each column with a single task: a_{ij} is equal to one if y_j refers to an assignment for task i ; otherwise, it is equal to zero. $B \in \mathbb{B}^{|R| \times |\Omega|}$ represents resource usage tuples: b_{rj} is one iff the given combination $r \in R$ of individual resources is used in the assignment y_j . Then, we build on the following integer programming (IP) formulation:

$$z = \min \left\{ \sum_{j \in \Omega} c_j y_j : \mathbf{y} \in \mathcal{P}_{\text{gspp}} \cap \mathbb{B}^{|\Omega|} \right\}, \quad (1)$$

where $\mathcal{P}_{\text{gspp}}$ denotes the polyhedral region defined by:

$$\sum_{j \in \Omega} a_{ij} y_j = 1 \quad \forall i \in T \quad (2)$$

$$\sum_{j \in \Omega} b_{rj} y_j \leq 1 \quad \forall r \in R \quad (3)$$

$$y_j \leq 1 \quad \forall j \in \Omega \quad (4)$$

$$y_j \geq 0 \quad \forall j \in \Omega \quad (5)$$

Set partitioning constraints (2) ensure that all tasks are served by exactly one assignment, while set packing in (3) forbids overlapping of assignments in each resource combination slot. We note that, if incompatibilities among tasks reduce to pairwise relations, the latter class of inequalities could be replaced by any set packing relaxation, such as *edge inequalities*: $x_u + x_v \leq 1$, for each edge (u, v) in a conflict graph (Atamtürk et al., 2000). Constraints (4) and (5) correspond to the linear relaxation of the binary programming formulation.

This formulation has connections to different disciplines in combinatorial optimization. It is similar to some variations of the assignment problem (Pentico, 2007). It can be seen as the scheduling of jobs on parallel machines minimizing total processing time, as we illustrate in Section 4.1 with a problem studied by Lalla-Ruiz and Voß (2016b). And the formulation is also an instance of the mixed set covering, packing and partitioning problem, investigated by Kuo and Leung (2016), following a longer tradition of studying perfect and ideal 0–1 matrices. The authors perform the first polyhedral investigation of the mixed problem, and argue on its relevance and number of applications, notwithstanding the fact that it has drawn little attention in the literature so far. For the interested reader, we indicate in Section 6 more related problems and situate the above GSPP structure in the set partitioning literature.

Our contributions The previous formulation can have a huge (though polynomial) number of variables. This is in consonance with the compromise between: (i) the complexity in the

representation of each execution mode to a task, *i.e.* the level of details included, and (ii) the computing time to solve the resulting problem.

The main idea of this paper is to show that one can use interesting, combinatorial constructions over the variables to find lower bounds to (1). Moreover, since the bounds are purely combinatorial, we suggest that its computation might be faster than linear programming (LP) relaxation ones, although possibly weaker. Even if the combinatorial bounds prove to be weaker, we argue that the construction itself might be interesting as a building block, *e.g.* in algorithmic approaches which depend on the computation of dual bounds, as we illustrate with a variable probing method and a matheuristic in Section 3.

To summarize, the contributions of this paper are:

- We claim attention to the GSPP structure (1) – (5) as interesting in its own right, and that there exists a range of applications of it in the recent literature. Furthermore, the key ideas we present can be extended to accommodate application-specific details, as we illustrate in Section 4.
- In Section 2, we introduce two graph representations relaxing part of the assignments in Ω , and show that weighted matchings in each of them yield lower bounds to problem (1), and that one of the constructions is stronger than the other.
- From an algorithmic standpoint, the combinatorial constructions allow for an embedding in a model-based heuristic framework (or *matheuristic*), presented in Section 3. We argue that the algorithm could only be conceived because of the efficient constructions, as opposed to the corresponding time to solve LP relaxations.
- Preliminary computational experiments with that matheuristic, using benchmark instances from a logistics problem, indicate that it is able to find near-optimal solutions in reduced time (Section 5).

2 Matching relaxations of the GSPP

In this section, we present the main contributions of the paper.

For the sake of clearness, we highlight from the problem definition in the Introduction three conditions for our key ideas to work. First, we assume that each individual assignment y_j has a cost c_j which is independent of the assignments to other tasks, so that its contribution to the objective function is computable when assuming y_j is used in the solution. We also assume that it is possible to determine in constant time whether a selection of individual assignments is compatible. Finally, the GSPP structure we study is limited to formulations with a polynomial number of variables, such that the set Ω can be enumerated before solving the resulting IP. While this immediately rules out a series of applications (typically solved by column generation algorithms), we hope that our discussions and numerical results in the remainder of the paper could settle the relevance of the class of problems we concern.

We start by introducing two relaxations for the GSPP formulation, which yield dual bounds to the optimal value of the objective z in (1). Note that the most naïve approach would be to discard all the packing constraints in (3) and simply pick the cheapest individual assignment to each task, which would provide a most trivial lower bound. In the following, we aim to discard less of those constraints. We construct two simple, undirected graphs, representing a subset of the enumerated assignments. Throughout the text, we use the linear map $c : \Omega \rightarrow \mathbb{Q}$ from the space of assignments to their costs, such that $c(y_j) = c_j$.

We define the graph $G_1(T, E_1)$, with a vertex for each task. The set E_1 includes an edge (i, j) if the individual assignments of best cost for tasks i and j are not compatible with each other. Let c'_j denote the minimum cost assignment for task j ; that is, $c'_j = \min\{c(y_j) : y_j \in \Omega_j\}$. Analogously, let c''_j be the second minimum cost assignment for j . The cost $c_1(i, j)$ of an edge in G_1 is defined by the least difference among such costs, for the corresponding tasks i and j . That is: $c_1(i, j) \triangleq \min\{(c''_i - c'_i), (c''_j - c'_j)\}$. Then, the following bound on the cost of any feasible solution holds.

Theorem 1. *Let $M \subseteq E_1$ denote a maximum weighted matching in G_1 , and $w(M) = \sum_{e \in M} c_1(e)$ be its weight. Then $LB_1 \triangleq w(M) + \sum_{j \in T} c'_j$ is a lower bound to the optimal value z in (1).*

Proof. The selection of the best individual assignments for each task corresponds to relaxing all the constraints in (3). Therefore, this is a trivial lower bound to the cost of any feasible

solution, and amounts to $\sum_{j \in V} c'_j$.

Starting with the trivial selection of best individual assignments, the weight of an edge $(i, j) \in E_1$ corresponds to the minimum cost increase due to exchanging one such assignment for the second best. Clearly, this new pair of assignments for tasks i and j can still be infeasible, but the sum of their costs is a lower bound to the cost of any compatible assignment for these tasks.

Note that we cannot imply that the accumulated costs of edges incident to a same vertex are necessary, because the graph does not provide information about which of the extremes of an edge assumes the second best assignment; it is even possible that, following such an exchange, other edges might not exist. However, one can consider any matching in G_1 , corresponding to disjoint pairs of tasks whose best assignments are not compatible. Therefore, the weight of any matching is a required cost increase over $\sum_{j \in T} c'_j$, implied by the pair-wise incompatibility of the corresponding individual assignments. In particular, a maximum weighted matching corresponds to the strongest such bound in G_1 . \square

Our second dual bound strengthens the information on the cost of compatible assignments between pairs of tasks. Let $G_2(T, E_2)$ denote a complete graph, with a vertex for each task. Define the cost $c_2(i, j)$ of an edge in E_2 as the cheapest compatible assignments for tasks i and j , that is: $c_2(i, j) \triangleq \min\{c(y_i) + c(y_j) : y_i \in \Omega_i, y_j \in \Omega_j, y_i \text{ and } y_j \text{ are compatible}\}$. Then, we have the following result.

Theorem 2. *Let $M \subseteq E_2$ be a maximum weighted matching in G_2 . Then, $LB_2 \triangleq \sum_{e \in M} c_2(e)$ is a lower bound to the optimal value z in (1).*

Proof. The weight of a single edge $(i, j) \in E_2$ is the sum of the minimum cost assignments for tasks i and j , complying with the compatibility constraints *among them*. That is: these two assignments alone are compatible. A selection of edges not sharing a vertex (*i.e.* a matching) thus corresponds to pairing up tasks and determining their best compatible assignments, which is required in any solution satisfying packing constraints (3). Therefore, the weight of any matching in G_2 is a lower bound to the cost of a feasible solution, since this clearly relaxes constraints regarding the compatibility of unpaired tasks. A maximum weighted matching

thus provides the strongest such bound in G_2 . \square

Although this result holds for any number of tasks, it would be unnecessarily weaker for odd $|T|$, since some vertex would not be covered by the matching, thus not contributing to the lower bound. To circumvent this, in the case that $|T|$ is odd, we simply add to G_2 an artificial vertex s , with edges to every other vertex i , with costs $c_2(s, i) = \min\{c(y_i) : y_i \in \Omega_i\}$.

We conclude with a result on the relative strength of the bounds obtained in the two relaxations. Note that, in the simple case where all the best individual assignments are pairwise compatible with each other, we verify: (i) the graph G_1 has no edges, and the bound LB_1 corresponds to trivial bound $\sum_{j \in T} c'_j$; (ii) any perfect matching M in the graph G_2 has maximum weight, amounting to the sum of costs of the best individual assignments. Therefore, the bounds are equal: $LB_2 = \sum_{(i,j) \in M} c_2(i, j) = \sum_{(i,j) \in M} (c'_i + c'_j) = \sum_{u \in T} c'_u = LB_1$. We show below that the second bound is actually stronger than the first.

Theorem 3. *The lower bound attained from graph $G_2(T, E_2)$ is stronger than that from graph $G_1(T, E_1)$; i.e. for any given problem instance, $LB_2 \geq LB_1$ holds, and $LB_2 > LB_1$ for at least one case.*

Proof. First, we remark that specific cases where $LB_2 > LB_1$ are intuitive. It suffices to have a pair of vertices for which there are no compatible assignments employing the cheapest execution mode for one of them.

In the following, we suppose there were an instance where $LB_1 > LB_2$. We build a matching in G_2 with cost at least LB_1 , showing that the hypothesis is absurd. We assume without loss of generality that the number of vertices $|T|$ is even, since, as described above, we suggest including an artificial vertex in the corresponding graph instances with odd $|T|$ to get a stronger bound.

For any edge $(i, j) \in E_1$, we can compare the cost functions in G_1 and G_2 ; recall that the latter graph is complete. By definition, $c_1(i, j)$ corresponds to the minimum cost increase implied by the incompatibility of the best individual assignments for i and j , while $c_2(i, j)$ corresponds to the actual sum of the costs of the best compatible assignments. It follows

that:

$$c_2(i, j) \geq c'_i + c'_j + c_1(i, j) \quad (6)$$

Let $M_1 \subseteq E_1$ be a maximum weighted matching in G_1 , which thus yields the lower bound LB_1 from that graph. We define the analogous set of edges in G_2 as $M_2 = \{(i, j) \in E_2 : \text{there exists the edge } (i, j) \in M_1\}$. The set M_2 is a matching in G_2 , by construction.

We distinguish two cases. If M_1 is perfect, then M_2 is perfect as well since both graphs have the same vertex set. We can infer about their weights:

$$w(M_2) = \sum_{(i,j) \in M_2} c_2(i, j) \geq \sum_{(i,j) \in M_2} (c'_i + c'_j + c_1(i, j)) = \sum_{(i,j) \in M_1} c_1(i, j) + \sum_{u \in T} c'_u = LB_1, \quad (7)$$

where the first inequality holds by (6), and the second equality is true because the matchings are perfect.

If M_1 is not perfect, there are pairs of vertices (x, y) not covered by M_1 . By hypothesis, M_1 has maximum weight; hence $(x, y) \notin E_1$, *i.e.* the individual assignments of least cost for x and y are compatible. Therefore, the edge in G_2 corresponding to each such pair (x, y) has cost $c_2(x, y) = c'_x + c'_y$. We can extend M_2 to a perfect matching M'_2 in G_2 by arbitrarily connecting pairs of vertices not yet covered by M_2 . Let C denote the set of edges selected this way, such that $M'_2 \triangleq M_2 \cup C$, and $\sum_{(x,y) \in C} c_2(x, y) = \sum_{(x,y) \in C} c'_x + c'_y$. Then, analogously to the previous case, we have:

$$\begin{aligned} w(M'_2) &= \sum_{(i,j) \in M_2} c_2(i, j) + \sum_{(x,y) \in C} c_2(x, y) \\ &\geq \sum_{(i,j) \in M_2} (c'_i + c'_j + c_1(i, j)) + \sum_{(x,y) \in C} c_2(x, y) \\ &= \sum_{(i,j) \in M_2} c_1(i, j) + \sum_{u \in T} c'_u \\ &= LB_1, \end{aligned} \quad (8)$$

where the last equalities hold because M'_2 covers all vertices.

Therefore, the matchings in G_2 built in both cases (7) and (8) have weight at least LB_1 ,

providing a lower bound on LB_2 , which is defined as the maximum weight of a matching in G_2 . Since we start with a general input instance, the hypothesis that $LB_1 > LB_2$ could hold is absurd, and we always verify that $LB_2 \geq LB_1$. \square

3 Embedding the relaxation in a matheuristic algorithm

This section extends our key idea, the matching relaxation of the GSPP structure, into algorithmic results. First, we derive in Section 3.1 a preprocessing method to probe and discard variables that imply a suboptimal solution, as it is done in the work of Iris et al. (2015) in the context of a port logistics problem. Next, we present in Section 3.2 a matheuristic algorithm to find approximate solutions to the problem in reduced computational time.

3.1 Preprocessing method for variable reduction

The previous results yield lower bounds on the optimal value z of problem (1), and can also be extended to a preprocessing method. The goal is to fix at null value (or, equivalently, remove) a number of decision variables in the resulting model after the enumeration of feasible assignments, while preserving the optimal, exact solution.

It is worth remarking that, since this technique is applied prior to the model optimization, such a proposal can be integrated with any approach based on enumerating the variables of the GSPP formulation and solving the resulting model with an integer linear programming algorithm. This strategy has already been adopted by Iris et al. (2015), using lower bounds implied by probing the selection of a single assignment or a pair of assignments for two different tasks. In Section 5, we compare their method with the one we propose.

The next result assumes that an upper bound to z is available. First, we temporarily assume that a given assignment $y_k \in \Omega_k$ is fixed in the solution. We define the complete graph $G_{2,k}(T \setminus \{k\}, E_{2,k})$. The corresponding edge costs $c_{2,k}$ regard the best compatible assignments

for two given tasks, which are also compatible with y_k . That is:

$$c_{2,k}(i, j) = \min\{c(y_i) + c(y_j) : y_i \in \Omega_i, y_j \in \Omega_j, \\ y_i \text{ and } y_j \text{ are compatible with each other and with } y_k\}$$

Finally, we evaluate the increase on the lower bound LB_2 from Theorem (2) implied by fixing the assignment y_k : if the new lower bound exceeds a known upper bound, we conclude that this assignment cannot be part of an optimal solution. The result is summarized as follows.

Proposition 1. *Let $LB_{2,k}$ denote the lower bound from Theorem 2 determined over $G_{2,k}$. Given any upper bound UB to z in (1), if $c(y_k) + LB_{2,k} > UB$, then there is no optimal solution which includes the assignment $y_k \in \Omega_k$, and the corresponding variable can be removed from the model.*

Therefore, we have an iterative algorithm for removing unnecessary variables in the model, while preserving all optimal solutions of the problem. For each feasible assignment in the GSPP formulation, one need only evaluate the new lower bound as depicted above.

Note that an analogous method could be derived from Theorem 1. Nevertheless, it follows immediately from Theorem 3 that it cannot be stronger, *i.e.* it cannot remove a variable which the result in Proposition 1 does not.

3.2 Combinatorial ranking matheuristic

We introduce next an algorithm belonging to the class of matheuristics, or model-based heuristics, which integrate heuristics and mathematical programming methods (Maniezzo et al., 2010; Ribeiro and Maniezzo, 2015). Specifically, we employ the combinatorial relaxation information to obtain a reduced model, which is optimized next with an integer linear programming solver. We remark that previous strategies in the matheuristics literature include solving a reduced model, *e.g.* after the heuristic removal of variables (Fanjul-Peyro and Ruiz, 2011; Stefanello et al., 2015). Even in the context of a subproblem of the logistics application that we use to illustrate our ideas (see Section 4.3), Mauri et al. (2008) present an evolutionary approach to generate columns using dual values in the LP relaxation as a fitness measure,

and [Lalla-Ruiz and Voß \(2016a\)](#) employ an exact solver to partially optimize components of a previous solution using the POPMUSIC metaheuristic.

First, note that every solution to the GSPP formulation consists of only $|T| \ll |\Omega|$ assignments. One could wonder if there would be a fast method for classifying variables, such that high quality solutions could be consistently achieved using only a fraction of the best ranked variables. In this context, we discard the optimality certificate, and seek a high quality solution in reduced computation time.

The core of the method we propose is depicted in Algorithm 1, which ranks and selects a subset Ω_F of variables from the GSPP formulation. The selection builds on the combinatorial bound $LB_{2,k}$ from Proposition 1. The bound is denoted by Δ in the algorithm, and computed on the loop starting at line 3. The set of selected variables Ω_F corresponds to a subset of the polyhedron $\mathcal{P}_{\text{gspp}}$, and optimizing over it provides an upper bound $\bar{z} \geq z$ to the original problem (1). As we indicate in preliminary computational results (Section 5), this bound can match the optimal value for benchmark instances of an application in port logistics even when using a relatively small fraction of variables.

The algorithm parameters are as follows.

σ : the percentage of the best ranked variables to include in the final model;

μ : the minimum number of variables corresponding to each task, which the algorithm should ensure (when available) in the final model.

The latter parameter μ ensures that each task has a number of assignment options (as selected in the loop starting at line 13), while σ controls the selection of variables among those implying the best dual bounds (loop starting at line 8). It would be natural to consider algorithm variations, *e.g.* selecting an exact number of variables, or performing a statistical study of the parameters. Both tasks could be approached in future work.

We remark that, since approaching a problem with this algorithm gives up on the optimality certificate, it is not necessary to reach a null duality gap in the resulting model to end the algorithm. Such a strategy may be interesting to meet runtime requirements in challenging applications.

Algorithm 1: combinatorial ranking

Input : initial set of variables Ω , number of tasks $|T|$
Output : set Ω_F of variables selected for the final model
Parameters: minimum percentage σ of the best ranked variables, minimum number of variables per task μ

```
1  $\Omega_F \leftarrow \emptyset$ 
2  $\Delta(k) \leftarrow 0$  for each  $y_k \in \Omega$ 
3 foreach  $y_k \in \Omega$  do
4   Let  $G_{2,k}$  be the graph of compatible assignments, defined in Proposition 1
5   Let  $M$  be a maximum weighted matching in  $G_{2,k}$ 
6   // lower bound implied by using this variable; see Prop. 1
7    $\Delta(k) \leftarrow c(y_k) + \sum_{e \in M} c_2(e)$ 
8 Let  $L$  denote the list of variables in  $\Omega$ , ordered by increasing values of  $\Delta$ 
9 while  $|\Omega_F|/|\Omega| < \sigma$  do
10   Let  $d$  be the least  $\Delta$  value of a variable in  $L$ 
11   foreach variable  $y_k \in L$  with  $\Delta(k) = d$  do
12      $\Omega_F \leftarrow \Omega_F \cup \{y_k\}$ 
13      $L \leftarrow L \setminus \{y_k\}$ 
14 foreach task  $i = 1, \dots, |T|$  do
15   while  $|\{y \in \Omega_F : y \text{ is an assignment for } i\}| < \mu$  and  $L$  contains a variable referring to  $i$  do
16     Let  $y_i \in L$  be a variable referring to  $i$ , of least  $\Delta$ 
17      $\Omega_F \leftarrow \Omega_F \cup \{y_i\}$ 
18      $L \leftarrow L \setminus \{y_i\}$ 
19 return  $\Omega_F$ 
```

We argue that Algorithm 1 is only useful because of the efficient combinatorial bound, as opposed to the corresponding time to solve LP relaxations in the loop starting at line 3. That is, instead of solving as many LP problems as $|\Omega|$ to determine the lower bounds Δ implied by fixing each variable, we reduce the computational effort to $(|\Omega|$ iterations of) the construction of graph $G_{2,k}$ and the solution of a weighted matching problem on a graph with just $|T|$ vertices.

In Section 5 we present the results of a successful computational experience this algorithm, using benchmark instances from a maritime logistics application. Concerning the execution time matter, even in the most difficult set of instances, solutions within 4% of the optimal value

are provided, with an average of 30% of the time required by methods recently introduced in the literature. On the other hand, for the sake of illustration, even if it took 1 second to build and solve the corresponding LP relaxation, a medium-size instance with 100.000 assignments would already take over a day to finish.

4 Sample applications

In this section we aim to give straightforward examples from the literature, in which relevant applications are formulated as different generalizations of a set partitioning problem. In each case, we give a brief problem definition, transcribe its IP formulation from the literature, and show at intuition level that they are amenable to the combinatorial constructions from Sections 2 and 3.

The application in Section 4.1 is a job scheduling problem, whose formulation is in direct correspondence with the GSPP structure that we present in the Introduction. As for the applications in crew disruption management (Section 4.2) and port logistics (Section 4.3), we highlight the interesting possibility to translate application features into small extensions of the matching relaxation.

4.1 Job scheduling in parallel machines

Lalla-Ruiz and Voß (2016b) concern the following problem, denoted Parallel Machine Scheduling with Step Deteriorating Jobs. Suppose that m identical, parallel machines are available in a planning horizon h , and that n jobs are to be scheduled, each consuming a processing time varying among two constants, as follows. If the assigned starting time is on a deteriorating date d_i or before, the job takes a base processing time a_i ; if it starts after that date, it requires processing time $a_i + b_i$. For each job $i = 1, \dots, n$, constants a_i , b_i and d_i are input parameters. The objective function is to minimize the sum of the completion times of all jobs.

The base formulation used by those authors corresponds precisely to the GSPP structure in (1) – (5), as we describe next. The set of columns is also denoted by Ω . Each column

corresponds to a variable x_w , representing a feasible assignment to an individual job. Thus, an assignment for job w in this problem encodes: the particular machine in which it is processed, the starting and finishing times, and a fixed cost c_w . Any selection of individual assignments to different jobs is compatible if no machine is used at a same time slot by more than one job.

Lalla-Ruiz and Voß also define the index set P such that $|P| = m \times h$, corresponding to machine and time slot combinations; this corresponds to $k = 2$ resources in our base structure. Then, using binary coefficient matrices $A \in \mathbb{B}^{n \times |\Omega|}$ and $B \in \mathbb{B}^{|P| \times |\Omega|}$ defined likewise (2) and (3) to express partitioning and packing constraints, they give the following formulation:

$$\min \sum_{w \in \Omega} c_w x_w \quad (9)$$

subject to:

$$\sum_{w \in \Omega} A_{iw} x_w = 1 \quad i = 1, \dots, n \quad (10)$$

$$\sum_{w \in \Omega} B_{pw} x_w \leq 1 \quad \forall p \in P \quad (11)$$

$$x_w \in \{0, 1\} \quad \forall w \in \Omega \quad (12)$$

We thus conclude this is trivially equivalent to the GSPP structure under investigation in this paper. Therefore, all among the combinatorial constructions of graph G_2 and the bound LB_2 (Theorem 2), the variable probing test (Proposition 1), and the matheuristic in Algorithm 1, carry over to this problem as presented before.

The next applications are more *interesting*, in the sense that the base GSPP structure provides only a relaxation of the formulated problem, making space for application-specific tweaks. Nevertheless, we highlight the fact that the base structure can, as it is, model such relevant topics as the job scheduling problem above.

4.2 Crew disruption management

Rezanova and Ryan (2010) investigate the train driver recovery problem, which aims to find the best assignment of tasks to replacement duties for train drivers, when the railway

operator has to recover from a disruption. In the occasion of internal or external failures (*e.g.* due to track conditions, accidents, or passenger delays), such that the slack time built into the timetable is not enough to restore the original plan, a recovery mission with the re-routing or cancelling of trains is performed. Dealing with the propagation of disruptions within the schedule makes the problem even more difficult for the operator.

The application itself includes a range of details which is beyond the scope of our discussion. While we limit the discussion below to the optimization of recovery duties, the elegant work of Rezanova and Ryan concerns several auxiliary issues around that central matter, and they evaluate their contributions in the context of real-life data from a Danish railway operator.

We start with a so-called *disruption neighbourhood*: let K denote a subset of train drivers whose duties include at least one disrupted train task within a given recovery period (which is our planning horizon). The remaining drivers keep their former duties, and are not included in the model. Also let N denote the set of all tasks originally assigned to drivers in K during the period, while P^k denotes the set of acceptable recovery duties for each driver $k \in K$. In this model, each individual assignment (such a recovery duty $p \in P^k$) encodes a subset of tasks in N and a fixed cost c_p^k , corresponding to a measure of its *unattractiveness* to driver k .

The goal of the train driver recovery problem is to find a selection of individual assignments to each driver in K (*i.e.* a feasible recovery duty) of least total cost, such that each task in N is covered exactly once. To this end, Rezanova and Ryan use binary decision variables x_p^k , set to one iff duty $p \in P^k$ is chosen. They also define a coefficient matrix analogous to B in the packing constraints (3) of the GSPP structure: let A , with $|N|$ lines and a column for each duty of each driver, be such that a_{ip}^k is one if task i is covered by duty $p \in P^k$, and zero otherwise. Now we can repeat their formulation, in (13) – (16) below.

The transformation of this formulation into the GSPP structure we consider allows for two possible relaxations. In each case, we describe what is relaxed, what corresponds to the packing constraints, and compare the combinatorial constructions and bounds.

$$\min \sum_{k \in K} \sum_{p \in P^k} c_p^k x_p^k \quad (13)$$

subject to:

$$\sum_{p \in P^k} x_p^k = 1 \quad \forall k \in K \quad (14)$$

$$\sum_{k \in K} \sum_{p \in P^k} a_{ip}^k x_p^k = 1 \quad \forall i \in N \quad (15)$$

$$x_p^k \in \{0, 1\} \quad \forall p \in P^k, \forall k \in K \quad (16)$$

A first, more natural, approach would be to relax equality constraints for tasks, allowing arbitrary $i \in N$ to remain uncovered in the solution. That is, we replace (15) by:

$$\sum_{k \in K} \sum_{p \in P^k} a_{ip}^k x_p^k \leq 1 \quad \forall i \in N \quad (17)$$

Then, formulation (13), (14), (16), (17) is a relaxation of the one by Rezanova and Ryan, and it can also be viewed as an instance of the base GSPP structure with a single shared resource ($k = 1$). In this case, the new, combinatorial relaxation yielding the bound in Theorem 2 consists of solving the weighted matching problem in a complete graph G_2 , with a vertex for each driver, such that the cost of an edge (i, j) corresponds to the cheapest, non-overlapping duties for the drivers i and j . Here, *non-overlapping* means only that no task is included in both duties of any pair of drivers. The bound thus sums up to pairing up drivers and finding cheapest pairwise compatible combinations; some tasks will likely remain uncovered, others might be included in duties of two or more (unpaired) drivers.

In the real world instances solved by those authors, we verify that $|K|$ and $|N|$ have the same order of magnitude. Then, the reunion of all duties in P^k should be larger than these: matrix A should have many more columns than lines. Therefore, the improvement of the bound described above, compared to simply removing all constraints in (15), could be negligible. Interestingly, since the formulation has two partitioning constraints, one can

conceive an alternative approach, relaxing instead the equality in constraints (14) into:

$$\sum_{p \in P^k} x_p^k \leq 1 \quad \forall k \in K \quad (18)$$

We therefore regard the integer program (13), (15), (16), (18) as a new relaxation, which requires that all tasks are covered exactly once, while allowing some drivers to remain idle. From this point, the new matching relaxation, attaining the bound from Theorem 2 for this application, builds on a complete graph, with a vertex for each task. Now, the weight of an edge (i, j) is determined by the cheapest pair of recovery duties covering both tasks i and j exactly once, in which no driver is assigned two different duties. That is, the compatibility oracle would return the cost of either: (i) a single duty for the same driver, including both tasks, or (ii) two duties, for different drivers, each covering one of the tasks but not the other. We remark that imparting more application details on the oracle decision for pairwise-compatible assignments could increase the edge weights and, consequently, strengthen the matching bound.

4.3 Port logistics

Our last sample application is also the subject matter of our preliminary computational evaluation, in the next section. The Berth Allocation and Quay Crane Assignment Problem (BACAP) aims to allocate a berthing time, a position in the quay, and a number of quay cranes (QCs) for arriving vessels in a seaport container terminal. Feasible assignments in the BACAP need to fulfil requirements on desired berthing period and position, and an agreement on the use of QCs. General reviews and a taxonomy to compare related work can be found in the surveys of [Stahlbock and Voß \(2008\)](#), [Bierwirth and Meisel \(2010\)](#), and [Bierwirth and Meisel \(2015\)](#).

We follow the presentation by [Iris et al. \(2015\)](#), who give a precise description of the application characteristics, and places their model for the BACAP in a convincing place in related literature. Let V be the set of vessels, T be the set of time slots in the planning horizon, L be the set of berthing positions in the quay, and Q be the number of available

QCs. Also define the set of berthing time/position combinations $P = T \times L$; this corresponds to $k = 2$ shared resources in our base GSPP structure. They define the set Ω as we have used throughout this paper: the complete set of feasible individual allocations. In this case, each element in Ω encodes a position in the quay, time slots and a number of cranes to serve a given vessel, besides its cost c_j , which depend on QC usage, deviations from parameters on the desired position on the quay and expected starting and finishing times for the service.

Binary decision variables $y \in \mathbb{B}^{|\Omega|}$ indicate which individual assignments are used in the solution. Finally, the coefficient matrices are in accordance with those in our description of the base GSPP, as we describe next. $A \in \mathbb{B}^{|V| \times |\Omega|}$ associates each column with a single vessel: a_{ij} is equal to one if column j refers to an assignment for vessel i ; otherwise, it is equal to zero. $B \in \mathbb{B}^{|P| \times |\Omega|}$ represents berths as combinations of time intervals and quay positions: b_{pj} is one iff the given pair of (time, space) positions corresponding to $p \in P$ is used in the assignment y_j . An element of $Q \in \mathbb{Z}^{|T| \times |\Omega|}$ determines how many QCs are used by y_j in time period t . Then, the BACAP formulation described by [Iris et al. \(2015\)](#) corresponds to:

$$\min \sum_{j \in \Omega} c_j y_j \quad (19)$$

subject to:

$$\sum_{j \in \Omega} a_{ij} y_j = 1 \quad \forall i \in V \quad (20)$$

$$\sum_{j \in \Omega} b_{pj} y_j \leq 1 \quad \forall p \in P \quad (21)$$

$$\sum_{j \in \Omega} q_{tj} y_j \leq Q \quad \forall t \in T \quad (22)$$

$$y_j \in \{0, 1\} \quad \forall j \in \Omega \quad (23)$$

Set partition constraints (20) ensure that all vessels are served by exactly one assignment, while set packing in (21) forbid overlapping of vessel assignments in each single time/space slot. Inequalities (22) guarantee that QCs availability in the terminal is respected.

It is clear, then, that one need only relax the inequalities (22) on the QC availability bound to view the formulation by *Iris et al.* as an instance of the GSPP structure we concern.

The matching relaxation from Theorem 2 can thus be determined on a complete graph, with a vertex for each vessel, where the cost of an edge amounts to the weight of a cheapest, compatible pair of assignments to the corresponding vessels.

Finally, we remark that two assignments for different vessels in the BACAP are denoted *compatible* if they have no overlap in berthing time and space. Equivalently, representing the two assignments in a Cartesian plane (with time and space coordinates), they are compatible iff the corresponding rectangles do not intersect each other. We can further tighten the definition of compatibility in this application by limiting the combined number of cranes used by two given assignments to the maximum available in the quay.

5 A computational case study

The goals of the computational evaluation we present are twofold. First, we want to compare the linear programming and the matching relaxations to evaluate the trade-off between the strength of the bound and the time to compute it. Second, we seek to assess the strength of the solutions provided by the matheuristic we introduced. Toward these ends, we have implemented a series of algorithms concerning the last application we described, the Berth Allocation and Quay Crane Assignment Problem (BACAP). We consider the same benchmark instances used by Meisel and Bierwirth (2009) and Iris et al. (2015) to evaluate the efficiency of the proposed algorithms. There are ten instances of three different sizes, with 20 (small), 30 (medium) and 40 (large) vessels.

All algorithms were implemented in C++ language using Gurobi solver version 6.5. To compute maximum weighted matchings, with the blossom shrinking algorithm by Edmonds (Edmonds, 1965), we use the efficient implementation available in the open source Library for Efficient Modeling and Optimization in Networks (LEMON) (Dezső et al., 2011). The time complexity of that implementation is $O(mn \log n)$ in the worst case, where n is the number of graph vertices and m is the number of edges.

All experiments were run in a machine with an Intel Core i7 4790K (4.00 GHz) CPU and 16GB of RAM. The solver runtime used in the matheuristic experiments was limited to

1800 seconds. The experiments corresponding to the techniques proposed by [Iris et al. \(2015\)](#) were not time limited. It is important to highlight that all results we present concerning the work by those authors were evaluated with our own implementation of their algorithms. Moreover, the numbers concerning their results may have slight variations when compared to the original work, which we have concluded to be explained by numeric precision matters. Also, the number of execution threads allowed for Gurobi to solve the model corresponding to that work is set to 2 in the case of instances with 20 or 30 vessels, while a single thread is allowed to solve instances with 40 vessels. Since the Gurobi solver needs copies of the complete model for each execution thread, we verified improved performance using this setting because no virtual memory is needed.

First, we compare the lower bounds and the runtime to compute the combinatorial relaxation, against the LP relaxation. Table 1 presents the corresponding bounds in columns 3 and 4. Column 5 indicates the percentual difference of the combinatorial bound with respect to the LP relaxation one. Columns 6 and 7 present the execution time required by each method.

The combinatorial bound can match the LP relaxation value for two instances, but it is consistently weaker and the difference also grows with the input size. Nevertheless, while building the model and solving its LP relaxation consumes significant time, building the graphs representing compatible assignment and solving the corresponding weighted matching problem is performed very fast, in comparison.

One can thus argue that, when choosing a lower bound to use in the variable reduction technique or in the matheuristic algorithm, the computational performance of the matching relaxation is the crucial factor. Since a lower bound must be computed after probing each variable, and even the smallest instances have tens of thousands of variables, the runtime of the LP relaxation becomes prohibitive in this context.

Next, we executed different experiments to evaluate the matheuristic algorithm. We stress again that the matheuristic approach waives the optimality certificate of a solution in the sake of a reduced computation time. In the following, we verify this outcome and assess the strength of the solutions attained, in comparison with the exact method in the literature.

Several parameter combinations were tested, with $\sigma \in \{0.0, 0.1, 0.2, 0.3\}$ and $\mu \in$

Table 1: Results attained with the combinatorial and the linear programming relaxations

Instance		Lower Bounds			Time (s)	
$ V $	ID	LP relaxation	Matching relaxation	Difference (%)	LP relaxation	Matching relaxation
20	1	885.0	698.0	21.1%	5.61	0.01
	2	562.0	562.0	0.0%	0.16	0.00
	3	816.5	646.0	20.9%	6.69	0.02
	4	762.0	620.0	18.6%	7.37	0.02
	5	592.0	516.0	12.8%	1.62	0.01
	6	592.0	592.0	0.0%	1.23	0.00
	7	722.0	646.0	10.5%	2.77	0.00
	8	582.0	532.0	8.6%	4.02	0.00
	9	782.0	620.0	20.7%	6.47	0.01
	10	930.7	700.0	24.8%	9.34	0.02
$\overline{\text{LBgap}}_{I_{20}}$				13.8%		
30	11	1,408.8	922.0	34.6%	29.98	0.06
	12	891.3	800.0	10.2%	5.23	0.01
	13	1,091.7	894.0	18.1%	7.64	0.02
	14	1,036.5	880.0	15.1%	12.19	0.02
	15	1,600.1	1,046.0	34.6%	36.58	0.09
	16	1,137.0	1,008.0	11.3%	7.56	0.01
	17	1,084.0	894.0	17.5%	9.07	0.02
	18	1,245.0	860.0	30.9%	31.71	0.07
	19	1,705.0	1,052.0	38.3%	44.07	0.08
	20	1,354.4	1,008.0	25.6%	22.13	0.03
$\overline{\text{LBgap}}_{I_{30}}$				23.6%		
40	21	2,058.8	1,150.0	44.1%	94.29	0.22
	22	1,680.0	1,288.0	23.3%	36.68	0.12
	23	2,380.7	1,250.0	47.5%	102.59	0.28
	24	2,727.0	1,544.0	43.4%	118.38	0.38
	25	1,559.9	1,102.0	29.4%	44.00	0.13
	26	2,364.1	1,294.0	45.3%	97.34	0.31
	27	1,965.8	1,222.0	37.8%	58.83	0.18
	28	2,533.3	1,412.0	44.3%	122.40	0.41
	29	2,071.8	1,404.0	32.2%	65.54	0.16
	30	1,872.5	1,240.0	33.8%	73.21	0.21
$\overline{\text{LBgap}}_{I_{40}}$				38.1%		

$\overline{\text{LBgap}}_{I_{|V|}}$ stands for the arithmetic mean of the percentual difference in the bounds for instances with $|V|$ vessels.

$\{500, 1000, 1500, 2000\}$. Preliminary evaluations with $\mu = 0$ and different choices for σ had a poor performance, leading to such an extreme reduction that the resulting model was infeasible for at least one instance, *i.e.* the corresponding polyhedron does not include a single integer point. This is important to justify the final step in our algorithm (requiring at least μ

variables corresponding to each vessel). The four parameter combinations with $\sigma \in \{0.0, 0.1\}$ and $\mu \in \{500, 1000\}$ also led to an infeasible model for at least one instance. The remaining combinations always found an integer feasible solution.

We report in Table 2 the results with the configuration $(\sigma = 0.1, \mu = 2000)$, which yields the least average GAP_{OPT} , between the best primal solutions and the known optima. For each instance, we present the optimal solution value z , the percentage of variables after the matheuristic filter (Ω_F), followed by the cost \bar{z} and the total GAP_{OPT} (from the optimal value z) of the best solution found through the matheuristic. In the next columns, we present the partial runtime of applying our variable reduction technique (T_C) and the one by Iris et al. (2015) (T_I), as well as the total runtime (*i.e.* the sum of the times for the reduction and for solving the mathematical model using Gurobi): T_{C_F} and T_{I_F} , respectively. Lastly, we show the runtime improvement of our proposal (T_E) compared to the one from the literature.

It can be seen from these results that the proposed methodology is able to find the known optimal solution in 83% of instances, while the gap is below 4% for those solutions which are not optimal. Note that the runtime efficiency of our methodology is inferior to the literature in half of the instances with 20 vessels and in one of the medium instances. Nevertheless, these results do not have a significant impact because the corresponding execution times are at most 44 seconds, and the time difference between the methods does not exceed 2 seconds. For example, while instance 2 is solved instantaneously by the technique from the literature, it spends 1.09 seconds using the matheuristic. Therefore, our proposal loses on the average efficiency for small instances.

On the other hand, observing the results for medium and large instances in the benchmark, it is clear that the matheuristic times are significantly smaller than those from the exact method presented by Iris et al. (2015). The known optimal solution is found in all instances with 30 vessels, spending on average 59% of the time spent by the exact method. For more difficult instances, with 40 vessels, the optimal solution is obtained in half of the cases, while very good solutions can be achieved using about 30% of the time consumed by the baseline method.

Considering instance 23, for example, the optimal solution is obtained 8 times faster by the

Table 2: Results concerning quality of solutions and runtime

Instance		Matheuristic results				Runtime				Efficiency
V	ID	z	$\Omega_F(\%)$	\bar{z}	Gap _{OPT} (%)	Matheuristic		Iris et al. (2015)		$T_E(\%)$
						T_C (s)	T_{CF} (s)	T_I (s)	T_{IF} (s)	
20	1	89.00	60.84	89.00	0.00	32.44	56.65	14.84	64.28	88.13
	2	56.20	100.00	56.20	0.00	1.01	1.09	0.09	0.16	681.25
	3	85.70	59.09	85.70	0.00	9.63	73.19	0.79	114.73	63.79
	4	81.80	47.86	81.80	0.00	8.58	39.24	2.22	60.22	65.16
	5	59.20	99.13	59.20	0.00	2.62	7.99	0.16	7.59	105.27
	6	59.20	100.00	59.20	0.00	1.53	6.73	0.16	5.09	132.22
	7	75.20	94.32	75.20	0.00	6.42	21.33	2.56	19.92	107.08
	8	61.40	73.20	61.40	0.00	5.39	22.10	0.95	21.41	103.22
	9	79.00	52.22	79.00	0.00	16.37	45.45	3.77	54.32	83.67
	10	101.00	45.56	101.00	0.00	26.05	76.27	3.13	214.43	35.57
									$\bar{T}_{E_{20}}$	101.34
30	11	143.20	31.43	143.20	0.00	60.66	186.16	7.64	421.91	44.12
	12	92.00	87.55	92.00	0.00	13.59	43.93	0.85	43.67	100.60
	13	110.00	66.45	110.00	0.00	24.53	64.00	3.01	85.90	74.51
	14	107.40	50.85	107.40	0.00	28.50	111.48	6.73	164.24	67.88
	15	168.40	26.53	168.40	0.00	70.46	236.08	10.64	665.77	35.46
	16	121.60	72.09	121.60	0.00	57.76	144.80	22.03	153.33	94.44
	17	109.40	61.39	109.40	0.00	28.97	73.98	4.30	85.56	86.47
	18	135.00	28.24	135.00	0.00	56.29	253.90	10.72	651.13	38.99
	19	176.20	23.18	176.20	0.00	86.51	252.40	27.41	737.74	34.21
	20	139.80	38.27	139.80	0.00	70.84	204.72	28.60	346.50	59.08
									$\bar{T}_{E_{30}}$	59.01
40	21	246.80	16.20	252.60	2.30	235.29	2035.52	20.90	9237.00	22.04
	22	178.40	30.14	178.40	0.00	171.43	449.33	45.42	694.91	64.66
	23	266.30	15.39	266.30	0.00	322.13	1771.94	131.24	14516.88	12.21
	24	307.00	15.39	318.20	3.52	1788.03	3588.11	538.36	11071.74	32.41
	25	164.60	28.63	164.60	0.00	105.97	457.75	8.21	1049.86	43.60
	26	258.20	16.31	261.90	1.41	274.46	2074.55	89.55	4337.52	47.83
	27	205.40	19.92	205.40	0.00	171.92	454.28	17.81	1520.09	29.89
	28	294.20	16.04	301.30	2.36	518.37	2318.46	66.65	42988.26	5.39
	29	227.60	19.96	229.70	0.91	680.40	2480.52	567.79	2563.04	96.78
	30	210.10	18.57	210.10	0.00	223.22	1470.65	46.59	4354.89	33.77
									$\bar{T}_{E_{40}}$	30.14

$\Omega_F = \frac{\Omega_M}{\Omega_3}$; Ω_M denotes the number of variables after the matheuristic filter; $T_E = \frac{T_{CF}}{T_{IF}}$; $\bar{T}_{E_{|V|}}$ corresponds to the geometric mean of T_E for instances with $|V|$ vessels.

matheuristic algorithm. In instance 28, the matheuristic consumes less than 6% of literature algorithm runtime to find a solution within less than 3% of the optimal value. These results suggest that classifying the variables with the combinatorial bound is an effective selection

criterion: the model size decreases drastically, the solution runtime reduces consistently in instances with more challenging dimensions, while keeping solutions up to 4% of the optimum.

6 Further related work and concluding remarks

The first paragraph in Section 1 of a classic review by [Balas and Padberg \(1976\)](#) reads:

“Among all special structures in (pure) integer programming, there are three which have the most wide-spread applications: set partitioning, set covering and the traveling salesman (or minimum length Hamiltonian cycle) problem; and if we were to rank the three, set partitioning would be a likely candidate for number one.”

We remark that, over forty years later, the set partitioning problem is still a central structure in integer programming, as illustrated by the several interesting applications mentioned throughout this paper. That early work of Balas and Padberg is a thorough introduction to the concepts and problems related to set partitioning, as well as the main polyhedral results and exact algorithms up to that date. While surveying the literature on this fundamental problem is beyond the scope of our paper, we still mention particular works that are more directly related to ours or particularly inspiring.

The awarded thesis by [Borndörfer \(1998\)](#) provides an extended review of polyhedral and algorithmic aspects related to the underlying set packing and covering relaxations. It also documents the components and implementation issues of a branch and cut algorithm for set partitioning, extending in numerous directions the approach of [Hoffman and Padberg \(1993\)](#) to an application in airline crew scheduling. Finally, it describes a real world application of vehicle routing in the context of public transportation for handicapped people, building on a decomposition of the problem in two steps, both of which require optimizing over a set partitioning structure. In the next two paragraphs we highlight the remarkable effort of that author in extending the results on polytopes with more substantial body of knowledge to larger classes of problems.

Borndörfer and Weismantel (2001) present an affine transformation technique (*aggregation*) to leverage cutting planes from combinatorial relaxations of an IP formulation. Their polyhedral investigation stems from a generalization of projection to introduce an algorithmic approach. A so-called *aggregation scheme* allows one to transfer (*expand*) known classes of valid inequalities in a suitable projection space (*e.g.* the polytope of another combinatorial structure) to the original polyhedron representing an IP. Besides constructing interesting aggregation schemes leading to set packing and knapsack relaxations, the authors describe conditions under which the separation problem for the new classes of inequalities (valid) can be solved in polynomial time. Several of these results are further investigated and contrasted within the framework of disjunctive cuts by Letchford (2001).

Borndörfer (2004) also describes the combinatorial packing problem (CPP), investigating a structure closely related to the class of set partitioning problems that we study. The problem consists of solving a number of individual combinatorial optimization problems on a same ground set, such that no element is contained in the solution of more than one problem. A number of interesting problems can be subsumed as CPP examples, including minimum cost flows, steiner trees packing, and the generalized assignment problem.

Finally, we mention a few more problems that are closely related to the GSPP structure that we investigate here. In the work of Campello and Maculan (1987), the set partitioning problem is generalized by a single upper bound on positive linear combinations of the variables, instead of the set packing constraints of the GSPP we study. They also consider a different cardinality constraint, and propose a Lagrangean relaxation scheme to determine lower bounds. Another interesting Lagrangean approach worth mentioning is that of Cavalcante et al. (2008) to set partitioning. Their sophisticated relax-and-cut algorithm improves on the quality of previous known lower bounds, while also being competitive regarding time efficiency. We humbly believe that extending their approach to the GSPP structure we study and comparing combinatorial and Lagrangean bounds would be a rather interesting work. As for Fisher and Kedia (1990), the set partitioning structure is generalized by set covering constraints. The resulting problem is also interesting, properly formulating previous applications in the literature, and the authors describe dual heuristics to it.

The long tradition in studying the polyhedra associated with 0–1 matrices encompasses the GSPP structure that we consider. In particular, [Sebő \(1998\)](#) extended the theory of perfect matrices in set packing formulations, and ideal matrices in set covering, to characterize non-integral polyhedra in the mixed packing and covering problem (assuming the coefficient matrices satisfy those former conditions). A central structure in his work is that of an odd-hole graph. Almost twenty years later, [Kuo and Leung \(2016\)](#) start from that same graph structure to investigate the mixed set covering, packing and partitioning problem.

They derive the *mixed odd-hole inequality*, and show that its inclusion completely characterizes the polytope corresponding to the mixed problem when the coefficient matrix induces precisely that graph. Further classes of inequalities are introduced, and preliminary computational experiments indicate its effectiveness. As we remarked earlier, the base GSPP formulation that we study is an instance of the mixed problem of Kuo and Leung. Nevertheless, the purely combinatorial constructions that we devised may be extended to include different problem features that cannot be modelled by their formulation. That is precisely what allowed us to strengthen the compatibility criterion with the number of available cranes in the port logistics application of [Iris et al. \(2015\)](#).

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